The Duality between Interlinked Electric and Magnetic Circuits and the Formation of Transformer Equivalent Circuits

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ABSTRACT. When making calculations on a circuit, containing both electric impedances and transformers, it is frequently desirable to consider the transformers removed and the constraints they impose replaced by a rearrangement of the impedances connected to their terminals. Such " equivalent circuits " may not always be found ; the rules are here established for their formation, and also for checking, by inspection, whether the transformer constraints are removable in this way, in any particular case. It is shown that the equivalent electric circuit of a transformer, having any arrangement of magnetic paths, is derivable from its magnetic circuit by application of the topological principle of duality. This cannot be done if the magnetic circuit is non-planar, as in the case of a transformer possessing four or more windings with leakage couplings; a physically realizable circuit does not then exist.

Under certain conditions the principle may be applied in reverse and the impedances in a given electric circuit may be coupled together by a suitable transformer, so that the various current and voltage constraints are unaltered.

§ 1. INTRODUCTION

THE simple geometrical relation which exists between inverse, or *dual*, circuits (Russell 1904) of electric impedances, based on the interchange of junction-points and meshes, is well known, and has been shown by Cauer (1934) to be based on the topological principle of duality. It is intended in this paper to show that an identical relation of pattern exists between the magnetic circuit of a transformer and its "equivalent electric circuit". Furthermore, there exists a simple inverse relationship between the magnetic impedances of the transformer and the electric impedances in the equivalent circuit, in a manner analogous to the impedance relations in the inverse *electric* circuits referred to above. By way of illustration, figure 1 shows such a pair of electric inverse (or dual) circuits. Given the first circuit A, a point is marked inside every mesh a, b, ... and one external to the circuit k; these points, when joined together as shown by the dotted branch lines, form the junction points of the dual circuit pattern B. This dual circuit B now has impedances inserted in its branches Z_1, Z_2, Z_3, ... which are proportional to the admittances a_1, a_2, a_3, ... in corresponding (crossing) branches of the first circuit A, in the sense $Z_1 : Z_2 : Z_3 : \ldots = a_1 : a_2 : a_3 : \ldots$ or $Z_n = K^2 a_n$, where K is any real constant. This network transformation replaces mesh connections in one circuit by star-point connections in the other and vice versa. The element *kinds* must also be changed, thus : capacity C in 1st circuit is replaced by inductance L in 2nd circuit and vice versa ; resistance R is replaced by conductance G and vice versa. Corresponding impedance magnitudes in such a pair of circuits are thus reciprocally related. This type of dual relationship is reversible, and starting with a circuit such as B the dual circuit A would be derivable. Using the symbol $\rightleftharpoons$ to mean "is dual to", the relationships between such a pair of circuits may be summarized thus :

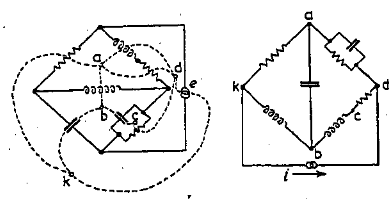
$$L \rightleftharpoons C; \qquad R \rightleftharpoons G. \qquad \ldots\ldots(1)$$

Again it will be found that voltages across branches in one circuit v_1, v_2, ... are proportional to the currents in corresponding (crossing) branches of the other, dual, circuit, so that

$$v \rightleftharpoons i. \qquad \ldots\ldots(2)$$

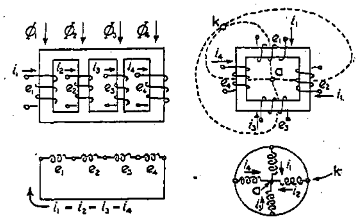
If one of the circuits forms a non-planar figure (so that it cannot be drawn on a flat sheet of paper without any one branch crossing another) there exists no dual circuit, except by the use of a special artifice given by Bloch (1946).

Analogous relationships will now be established for transformers and their equivalent circuits, taking for granted the rules for the setting up of dual mesh and branch patterns illustrated by Figure 1, and without employing the specialized notation and language of topology.



Circuit A. Circuit B.

Figure 1. Method of constructing dual circuits.



(a) (b)

Figure 2. Ideal transformers and the apparent connection of their windings into "equivalent circuits".
(a) mesh-type transformer;
(b) junction-type transformer.

§ 2. EQUIVALENT CIRCUITS OF IDEAL TRANSFORMERS

Figure 2(*a*) shows an ideal 4-limb transformer having windings with an arbitrary *equal number of turns*, N, wound on a core of infinite permeability. The limbs, which may have different cross-sections, carry fluxes Φ_1, Φ_2, ... which, meeting at a junction, must add up to zero; hence $\Phi_1+\Phi_2+\Phi_3+\Phi_4 \ldots = 0$. Differentiating and multiplying by N shows that the induced voltages must also add up to zero :

$$e_1+e_2+e_3+e_4+\ldots = 0. \qquad \ldots\ldots(3)$$

Furthermore, the M.M.F.s of the various limbs are in parallel and must be equal, since the transformer is ideal :

$$i_1=i_2=i_3=i_4=\ldots=i. \qquad \ldots\ldots(4)$$

These principles apply to ideal transformers having any number of parallel limbs carrying coils of equal turns.

From equations (3) and (4) the transformer coils may be considered to be connected in an *equivalent circuit* as shown in Figure 2(*a*); the various coil terminal voltages are in series and a common mesh current i flows. Such an arrangement is not a true "equivalent circuit" in that the impedances of the various coils are unknown since the original transformer was ideal. This circuit should, at this stage, be regarded merely as a possible pattern into which the various voltages and currents of the transformer windings may be arranged so that the same constraints between them apply as are represented by equations (3) and (4).

A second type of ideal transformer, shown in Figure 2 (*b*), has a series magnetic circuit; the equations for its currents and voltages are

$$i_1 + i_2 + i_3 + i_4 + \dots = 0 \qquad \dots\dots(5)$$

and

$$e_1 = e_2 = e_3 = e_4 = \dots = e \qquad \dots\dots(6)$$

so that the coils may again be imagined to be connected in an "equivalent circuit", as shown in the lower part of Figure 2 (*b*). Here the terminal voltages are all in parallel and the currents entering the junction point must total zero.*

It may be seen that these circuits can be formed by applying the rules for dual figures, explained in § 1, so that meshes in the first circuit are replaced by junction points in the second. In this present ideal case the equivalent electric circuit is the dual figure of the *magnetic* circuit, as has been illustrated by the superposed dotted circuit in Figure 2 (*b*).

The physically dual quantities are again those which lie on intersecting branches but bear no direct relation to the dual electric impedances of equations (1) and (2). Thus, *flux rate of change* is dual, here, to *voltage*, while *current* is dual to M.M.F. in the intersected limb.†

There again exists a constant of proportion between the quantities in such dual circuits. The duality of these electric and magnetic circuits is clearly reversible. Thus if we had started with the electric circuit we could have obtained the transformer magnetic circuit by drawing the dual geometric figures (shown dotted in Figure 2 (*b*)) and replacing voltage e by flux rate of change $d\Phi/dt$, and current i by M.M.F. M. Thus the duality may be expressed as

$$e \rightleftharpoons d\Phi/dt; \quad i \rightleftharpoons M. \qquad \dots\dots(7)$$

One further example is needed to illustrate the complete rules for ideal transformers, viz. a transformer containing both series and shunt limbs. The transformer shown in Figure 3 (*a*) is of such a type and, to make the case more general, two windings are shown on one of the limbs. All coils have N turns. With the fluxes as shown, the relation holds :

$$\Phi_5 = +(\Phi_1 + \Phi_2) = -(\Phi_3 + \Phi_4). \qquad \dots\dots(8)$$

The schematic magnetic circuit of M.M.F.s ($M_1 M_2 \dots$) and fluxes has been drawn in Figure 3 (*b*), which also shows the dual figure (dotted lines) drawn according to the rules of interchange of meshes and junction points. On the assumption that this figure gives correctly the "equivalent circuit" of the transformer, it has been redrawn in Figure 4 with the dual currents and voltages marked in. Here the new junction points a, b .... k have been identified.

* The directions of the currents in the transformer windings and in the equivalent circuit may be related, if the sense of the coil winding is taken into account. Thus in Figure 2, with the windings as shown, the currents in them are assumed positive in the directions marked ; in this case the currents in the equivalent circuits have been made consistent. It is similarly possible to relate the directions of flux rates-of-change and of voltages.

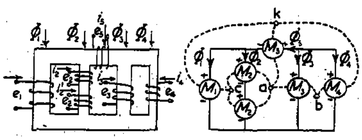
† An arbitrary set of rules could be applied for relating the *directions* of the currents and of the M.M.F.s in these two superimposed figures. For example, with reference to Figure 3, it could be stated that current direction is positive when it crosses a magnetic branch containing an M.M.F. which is positive on the left-hand side.

That this circuit is correct as regards the two coils on the limb carrying flux Φ_2 follows from the fact that the voltages of the coils are identical, so that they may be regarded as being in parallel with one another. Again, the two limbs carrying fluxes Φ_3 and Φ_4 are magnetically in parallel and their M.M.F.s must be equal, or $M_3 = M_4$; the dual electric branches, a, b and b, k, carry equal currents, being in ries.

However, the branch a, k must be shown to be equivalent to that dual limb on e transformer which carries the flux Φ_5. The voltage across this coil is $Nd\Phi_5/dt$, hich, from equation (8) is

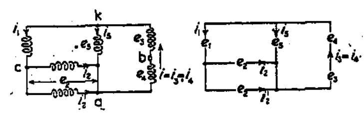
$$e_5 = Nd\Phi_5/dt = N(d\Phi_1/dt + d\Phi_2/dt) = -N(d\Phi_3/dt + d\Phi_4/dt) = (e_1 + e_2) = -(e_3 + e_4), \quad \ldots\ldots(9)$$

d this is seen to be consistent with the mesh arrangement of the equivalent rcuit in Figure 4.



(*a*) (*b*)

Figure 3. The dual of a multi-limb transformer.
(*a*) multi-winding ideal transformer;
(*b*) magnetic circuit (full lines) and the dual figure.



(*a*) (*b*)

Figure 4. (*a*) The "equivalent circuit" of the ideal transformer in Figure 3; (*b*) the current and voltage pattern.

Other examples may be treated in the same way. No matter how many limbs the transformer may have, or what may be the arrangement of the coils, their connection into an equivalent circuit of the type described may be derived by this routine method of drawing dual figures, provided that: (i) all coils have equal turns, N; (ii) the transformer magnetic circuit is a planar figure. Similar relations exist between these dual electric/magnetic quantities as between the ordinary electric circuit dual quantities (voltage and current) which have been represented by equation (2). The fluxes (and hence their rates of change) in the various limbs bear certain ratios to one another; the voltages in the (dual) equivalent electric circuit have the same ratios. That is, in the steady state,

$$j\omega\Phi_1 : j\omega\Phi_2 : j\omega\Phi_3 : \ldots . = e_1 : e_2 : e_3 : \ldots . \quad \ldots\ldots(10)$$

A similar relation between the M.M.F.s and currents may be written

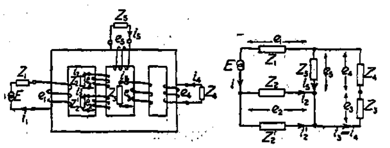
$$M_1 : M_2 : M_3 : \ldots . = i_1 : i_2 : i_3 : \ldots . \quad \ldots\ldots(11)$$

The question of magnetic and electric *impedance* duality analogous to (1) does not arise in these ideal transformers, but will be dealt with in §4.

§3. INCLUSION OF THE EXTERNAL CIRCUIT

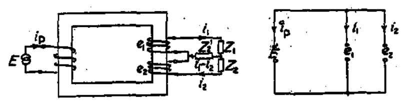
When a transformer forms a component part of a circuit it exercises certain constraints on the currents and voltages. These constraints arise by virtue of the magnetic couplings between the transformer windings to which the external circuit is connected. When the transformer is replaced by an equivalent circuit, this must be arranged so as to exercise the same constraints. In the case of the ideal transformer this equivalent circuit has been derived by applying rules of duality to the transformer magnetic circuit; the external loads and generators must not be included when these rules are applied, but must be imagined disconnected from the transformer windings and then, when the dual figure of the magnetic circuit has been formed, they may be reconnected to corresponding points on this dual circuit so as to support the same currents and voltages as originally.

For example, Figure 5 (*a*) illustrates an ideal transformer connected to a set of loads and a generator E. This particular transformer has already been used as an example (Figure 3) to illustrate the use of the duality rules, from which it has been shown that the various windings may be considered to form the equivalent circuit connections in Figure 4. This "circuit" is more correctly to be regarded as an arrangement of the various windings, currents and voltages into a pattern which is consistent with the constraints exercised by the transformer. It may alternatively be illustrated by a pattern such as that in Figure 4 (*b*). The various loads $Z_1, Z_2, \ldots$ and the generator E may now be connected in this pattern so as to support the correct voltages and currents. It will be noticed that the resulting circuit (Figure 5 (*b*)) contains no elements related to the transformer itself since, in this case, this is ideal and possesses no leakage inductance.



(*a*) (*b*)

Figure 5. A loaded ideal transformer and its equivalent circuit having identical current and voltage relations. (*a*) loaded transformer; (*b*) equivalent circuit.



(*a*) (*b*)

Figure 6. A transformer with interconnected windings and the failure of the "equivalent circuit".

It may sometimes be found that it is not possible to carry out this process as, for example, when there is an external impedance common to two windings. Thus Figure 6 (*a*) shows a transformer with two secondary windings having equal, induced voltages, e_1, and common connections between them. The two secondary currents are i_1 and i_2. The equivalent voltage and current "circuit", or pattern, formed by the dual figure of the magnetic circuit, is given by Figure 6 (*b*). In this pattern the two secondary voltages naturally appear in parallel and there is no way in which the external circuit Z_1, Z_2 and Z_2' may be connected so as to support the same currents and voltages as in the actual system. In general it is only when each winding is connected to a separate load or generator that connection of such loads in an equivalent circuit is guaranteed. The equivalent circuit pattern merely epresents one possible self-consistent arrangement of the transformer winding urrents and voltages; an entirely different set of constraints may apply to the external circuit. In any given example it is usually possible to check the existence of an equivalent circuit by forming the dual figure of the transformer magnetic circuit and attempting to reconnect the external electric circuit to this figure, paying due regard to directions of currents and M.M.F.S.

§ 4. THE PRACTICAL TRANSFORMER—ITS MAGNETIC CIRCUIT

In practice, transformer cores possess finite permeability, and their windings set up leakage fluxes. A practical, loaded, transformer may be regarded either as an electric circuit having a certain arrangement of voltages, currents and impedances or as a magnetic circuit of M.M.F.s, fluxes and reluctances. It is intended now to show that these circuits are duals, as was the case for ideal transformers.

Figure 7 (*a*) illustrates a simple two-winding transformer with primary and secondary leakage fluxes Φ'_p and Φ'_s and a finite core reluctance S, which may possibly include effects of air gaps. The schematic magnetic circuit is given in Figure 7 (*b*). S_p and S_s are the reluctances of the (air) paths of the primary and secondary leakage fluxes.

The current i_p flowing in the N-turn primary coil sets up an M.M.F. Ni_p, which should be regarded as the *driving* M.M.F. M_p. The induced current i_s in the secondary winding gives rise to *a response or back*-M.M.F., M_s. The resultant of these M.M.F.s sets up the working core flux Φ.

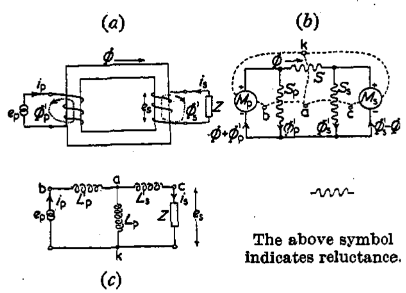


Figure 7. The practical transformer. Its magnetic circuit and (dual) equivalent circuit.
(*a*) transformer with leakage fluxes; (*b*) magnetic circuit; (*c*) equivalent circuit.

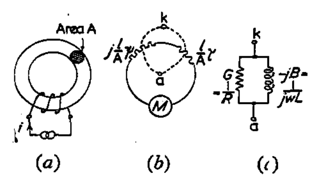


Figure 8. Equivalent circuit with lossy magnetic material.

The dual figure of this magnetic circuit (shown dotted) may now be formed according to the rules, but the question arises: what electric elements are dual to the reluctances? Reluctance causes a loss of M.M.F. proportional to the flux passing through it, $M = \Phi S$; in the (dual) equivalent circuit M.M.F. is replaced by its dual, current, and this must flow through *an electric admittance* to give a drop in voltage, dual to $d\Phi/dt$. This admittance is then the dual of reluctance, and furthermore it must be inductive.

This may be seen more clearly by dividing corresponding sides of equations (10) and (11):

$$\frac{M_1}{j\omega\Phi_1} : \frac{M_2}{j\omega\Phi_2} : \frac{M_3}{j\omega\Phi_3} : \ldots . = \frac{i_1}{e_1} : \frac{i_2}{e_2} : \frac{i_3}{e_3} \ldots .,$$

that is, the ratio between the various magnetic impedances is the same as the ratio between the corresponding electrical admittances. If the electrical elements be *inductances* this last equation may be written

$$\frac{M_1}{j\omega\Phi_1} : \frac{M_2}{j\omega\Phi_2} : \frac{M_3}{j\omega\Phi_3} : \ldots . = \frac{1}{j\omega L_1} : \frac{1}{j\omega L_2} : \frac{1}{j\omega L_3} : \ldots .,$$

so that *the duality relation is independent of frequency* :

$$S_1 : S_2 : S_3 : \ldots . = 1/L_1 : 1/L_2 : 1/L_3 : \ldots .$$

It is seen that once again there exists a reciprocal relationship between the dual physical elements, S_n and L_n, just as in the case of electrical duality, where the dual impedance elements Z_n and $1/a_n$ were reciprocals (§ 1), since

$$S_n = Ni_n/\Phi_n \quad \text{and} \quad L_n = N\Phi_n/i_n \quad \text{or} \quad S_n L_n = N^2. \qquad \ldots\ldots(12)$$

The equivalent circuit for the practical transformer, in the example, has been drawn in Figure 7 (*c*). This equivalent circuit is, of course, well known. The values of the inductance elements are given directly from the values of the dual magnetic reluctances from which they were derived : L_p is dual to S, and so $\propto \Phi/(i_p - i_s)$; L_p' is dual to S_p, and so $\propto \Phi_p'/i_p$; L_s' is dual to S_s, and so $\propto \Phi_s'/i_s$. Note that this equivalent circuit, obtained by the duality rules, is still correct if the secondary be open-circuited or short-circuited. In the first case $i_s = 0$ and the M.M.F. M_s vanishes; that is, it is magnetically *short-circuited*. In the second case $e_s = 0$ and the flux $(\Phi_s' - \Phi)$ vanishes; that is, $\Phi_s' = \Phi$ and the magnetic branch (containing M_s) is open-circuited.

The equivalent electric circuit of any other practical transformer is related to its magnetic circuit in this same dual way, whatever the arrangement of the limbs, provided that the magnetic circuit (including leakage flux paths) forms a planar figure.

§ 5. PRACTICAL TRANSFORMER—INCLUSION OF CORE LOSS

To be correct practically, the magnetic circuit of a transformer must include loss elements which are due to the core hysteresis and eddy currents. The values of such core losses and effective permeabilities are known for many common types of magnetic material (Macfadyen 1947).

The presence of hysteresis loss in a magnetic sample is indicated by the finite area of the hysteresis loop; the average loop is of such a shape as to necessitate a non-linear relationship between $\mathfrak{B}$ and $\mathfrak{H}$, but if the production of harmonics be ignored then, as far as fundamental components are concerned, the hysteresis loop may be regarded as a thin ellipse. This elliptical shape is actually approached in practice if the effects of eddy currents are appreciable. Thus both $\mathfrak{B}$ and $\mathfrak{H}$ may be taken as having sinusoidal waveforms slightly displaced in phase, a point of view which has been adopted by Macfadyen (1947) in the development of the idea of complex permeability.

It is convenient to use this idea here and to regard the magnetic circuit elements, reluctance and "magnetic loss", as complex magnetic impedances, $\vec{S}$.

The simplest magnetic circuit that may be taken for illustration is a ring having a mean magnetic path length l and a cross-section A (Figure 8 (*a*)). The reluctance of this core, $\vec{S}$, is related to the permeability, $\vec{\mu}$, by the expression $\vec{S} = (l/A)(1/\vec{\mu})$ amp-turns/weber (henries^{-1}) $= (l/A)(\tau + j\psi)$. These real and imaginary series components of reluctance are illustrated in the magnetic circuit of figure 8 (*b*).

If a coil of N turns be wound on this core, carrying a sine-wave current, i, and having an induced sine-wave voltage, e, the magnetic field-strength and flux-density become

$$\mathfrak{H} = Ni/l \text{ amp-turns/m.,} \qquad \ldots\ldots(13)$$

$$\mathfrak{B} = -j\vec{e}/\omega NA \text{ weber/m}^2, \qquad \ldots\ldots(14)$$

assuming that the $\mathfrak{B}/\mathfrak{H}$ loop is elliptical.

Dividing corresponding sides of (13) and (14) relates the magnetic reluctivity to the electric admittance : $1/\vec{\mu}=(\tau+j\psi)=jN^2\omega A/l\vec{Z}$ metres/henry. Thus an equivalent electric circuit may be found for the coil, shown in Figure 8 (*c*), which includes the effects of the complex reluctance $\vec{S}=(l/A)(\tau+j\psi)=jN^2\omega(\mathrm{G}-j\mathrm{B})$ amp-turns/weber (henries^{-1}). Here the electric circuit is given as an admittance $1/\vec{Z}=(\mathrm{G}-j\mathrm{B})$ shown in the figure as a resistance R in parallel with an inductance L. Comparison of the real and imaginary parts of the above equation for $\vec{S}$ shows that : (i) the *real part* of the reluctance $\vec{S}$ gives rise to a *susceptance* B, in the equivalent circuit: $l\tau/A \propto \mathrm{B}=1/\omega L$ (as has been shown already for the loss-less core); (ii) the imaginary part of the reluctance $\vec{S}$ gives rise to a *conductance* G, and $(l\psi/A) \propto \mathrm{G}\,(=1/R)$.

Notice that this equivalent circuit depends on angular frequency ω and so may be applied only to transformers working at a constant frequency.

This electric circuit, Figure 8 (*c*), is the dual of the magnetic circuit, Figure 8 (*b*); the two electric elements in parallel correspond to the two magnetic elements in series. However complicated the magnetic circuit of a transformer may be, such an equivalent electric circuit for each limb may be derived.

§ 6. THE MULTI-WINDING PRACTICAL TRANSFORMER

It is interesting to apply the method of duality to setting up the equivalent circuits of multi-winding transformers and to show that, in the general case, there can be no physical circuit if the transformer has four or more windings.

Figure 9 (*a*) shows a transformer with one primary and two secondary windings, assumed to be coupled not only by the main working flux Φ but also each to each by leakage fluxes. A possible fourth winding has been shown by dotted lines. In (*b*) the equivalent magnetic circuit is shown; here the M.M.F.s M_p, M_1, M_2, ... are those due to the currents in the primary and in the secondary windings. These various M.M.F.s are shown coupled by leakage reluctance paths which are denoted by suffixes : thus S_1 is a reluctance shunting M_1 whereas S_{12} couples M_1 and M_2 and so on. The addition of the fourth winding to the magnetic circuit is shown by the dotted lines ; a new secondary M.M.F. M_3 is coupled to the others by reluctances S_{123}, S_{23}, etc.

Considering the 2-secondary case only, the equivalent electric circuit may be constructed dually in the usual way, by marking in reference points in every magnetic mesh and joining up to form the dual circuit (Figure 9 (*c*)). The various inductances in this circuit are marked by suffixes corresponding to the various appropriate (dual) leakage reluctances, their respective magnitudes being given by equation (12).

In the case of three secondaries, however, such a dual equivalent electric circuit cannot be formed because the magnetic circuit is a non-planar figure. The case considered here includes every possible arrangement of mutual couplings of the windings by leakage fluxes; equivalent circuits of multi-winding transformers have, however, been developed using negative elements. (For example see Blume, Camilli, Boyajian and Montsinger, 1938.) That this figure must be non-planar follows from the topological fact that five or more points in a plane (i.e. the M.M.F. "terminals") cannot be joined each to each by lines (i.e. leakage reluctance paths) without at least one cross-over. Such an essential cross-over is marked * in Figure 9(*b*). This fact depends only on the *number* of M.M.F. "terminals" and not on how they are arranged in the magnetic circuit.

§7. TRANSFORMERS WITH UNEQUAL TURNS ON THEIR WINDINGS

If the principle of duality is to be applied to the construction of equivalent circuits for practical transformers, those cases cannot be excluded in which the turns ratios between the various windings are not unity.

The duality relationship which exists between the magnetic and the electric circuits of a transformer has been established in the preceding sections by starting with the ideal transformers of the "mesh type" and "junction type", illustrated

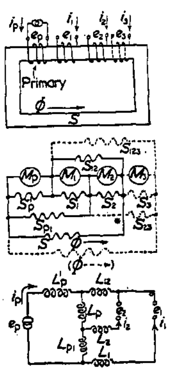


Figure 9. (*a*) A transformer with three windings; (*b*) its magnetic circuit; (*c*) the equivalent electric circuit and (dotted) the effect of adding a fourth winding.

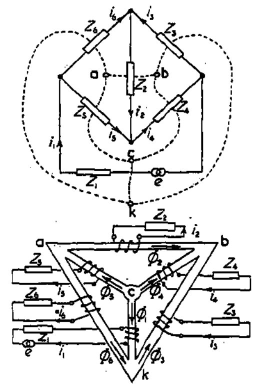


Figure 10. (*a*) A bridge circuit and (dotted) its dual figure; (*b*) the dual transformer and its loads.

in Figure 2; the equations (3), (4), (5) and (6), which give the current–voltage or M.M.F.–flux relations, involve the necessary condition that all the coils have an equal number of turns, N. If the coils have an unequal number of turns, these equations are modified. Thus if, in the mesh-type transformer, the limb fluxes, Φ_1, Φ_2, Φ_3, ... link with coils having, respectively, N_1, N_2, N_3, ... turns, then equations (4) and (5) modify to $N_1i_1 = N_2i_2 = N_3i_3 = \ldots.$, that is all limb M.M.F.S, being in parallel, are equal, and similarly $e_1/N_1 + e_2/N_2 + e_3/N_3 + \ldots. = 0$, since the sum total of the limb fluxes must be zero.

The above equations may be divided by n, the highest common factor of N_1 N_2, N_3, ..., giving

$$\frac{i_1N_1}{n} = \frac{i_2N_2}{n} = \frac{i_3N_3}{n} = \ldots. \quad \text{and} \quad \frac{e_1}{N_1/n} + \frac{e_2}{N_2/n} + \frac{e_3}{N_3/n} + \ldots. = 0.$$

These equations represent the voltage and current relations that exist in various "sections" of the individual coils, where one "section" is assumed to have n turns. Then, treating each section as a separate coil, the dual figure may be constructed and a nominally equivalent circuit to the transformer obtained. Such a circuit cannot, however, be the true equivalent circuit, because the n-turn sections into which the coil on a particular limb has been divided are electrically in series on the limb but are in parallel in this nominally equivalent circuit; the voltages across the transformer windings, which are proportional to the turns N_1, N_2, N_3, ..., do not appear in this equivalent circuit at all.

Again, a junction-type ideal transformer may be treated in a similar way, and in this case, too, no truly equivalent circuit can be derived.

In the case of practical transformers, with cores of finite permeability (possibly complex) and having leakage fluxes, true equivalent circuits consisting entirely of practical electric elements cannot be formed unless all windings have equal numbers of turns. However, by the inclusion of *ideal* transformers in these equivalent circuits the usefulness of the duality relationship may be extended to transformers with non-unity turns ratios. Thus if such a practical transformer has windings possessing unequal numbers of turns N_1, N_2, N_3 .... these may be reduced, in a theoretical way, to an equal number n by associating an ideal two-winding transformer with each separate winding; these various ideal transformers must then have turns ratios of N_1/n, N_2/n, ...

The equivalent electric circuit may now be found by the duality rules in the manner described in the earlier sections, though care must be taken to include the ideal transformers *as part of the external loads* and not to apply the duality rules to them.

§ 8. INVERSION OF THE MAGNETIC-ELECTRIC DUALITY PRINCIPLE

The duality principle has so far been applied to the production of an electric circuit equivalent to a transformer (i.e. a magnetic circuit) connected up to external loads and sources of power. As with other dual relationships, such, for example, as the electric circuit duality described in § 1, the procedure may be inverted and, given a planar circuit of purely electric impedances and sources of power, an *equivalent transformer* may be constructed. That is to say, these electrical impedance elements may be assembled on to a transformer in such a way that the constraints between the various currents and voltages are unchanged. That such an inversion of the process is possible has been shown in § 2 and expressed by equation (7).

By way of example, Figure 10 (a) shows a bridge circuit of any linear complex impedances Z_1, Z_2, Z_3, ... Z_6, carrying branch currents i_1, i_2, i_3, ... i_6. One source of power is included, for generality, in series with Z_1. The dual figure may be formed by marking reference points inside every mesh a, b, c, ... with one outside, k, and joining up by lines (shown dotted) intersecting every branch of the circuit. This dual figure gives the required arrangement (Figure 10 (b)) of the transformer limbs and windings which are loaded by the same impedances, Z_1, Z_2, ... Z_6. In this transformer the various limb fluxes are dual to the circuit branch voltages (more correctly, the flux rates of change if non-steady-state is considered) while M.M.F.s are dual to the branch currents in corresponding (crossing) branches.

Such a transformer is in general ideal, the core having infinite permeability and the number of turns on every winding being arbitrary, but equal. A practical transformer may only be derived *if the original circuit contains inductances* so that the various limbs, dual to the circuit branches, may possess finite reluctances.

Obviously, if the original electric circuit is a non-planar one, possessing cross-over branches, it has no dual figure and, therefore, cannot be assembled on a transformer in this way. However, part of the circuit may be planar, and those particular branches may be dealt with in the manner described and replaced by an equivalent transformer.

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An Investigation of the Dynamic Elastic Properties of Some High Polymers

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ABSTRACT. An apparatus for the investigation of the transmission of sound along filaments at frequencies between 1,000 and 6,000 cycles per second is described, both for unstrained specimens and whilst they are being elongated at a constant rate of increase of strain. Measurements of the dynamic elasticity and damping factors of filaments of polythene, neoprene and nylon have been obtained, and the correlation of these results with the molecular re-arrangements which take place during stretching is discussed.

§ 1. INTRODUCTION

THE properties of long chain polymers differ in many ways from materials composed of smaller molecular units and, in particular, the mechanical behaviour is considerably modified when the molecular chains become oriented. Thus the stress–strain curve for such materials is complicated by orientation produced on applying stresses. In general the elastic moduli are found to be much greater for highly oriented materials than for such materials in the isotropic state.

Also, with most of these substances if the stress is applied slowly, irreversible flow takes place, and the strain is a function not only of stress but of time.

The ordinary stress–strain curves of such materials do not, therefore, easily yield information about the molecular processes which take place during deformation.

Some workers have applied alternating stresses of various frequencies to high polymer specimens and measured the resulting strains in efforts to obtain more complete knowledge of the mechanical behaviour of these materials. Recently, measurements of sonic and ultrasonic velocities in fibres of various long chain materials have been made in the U.S.A. by Ballou and Silverman (1944) and Nolle (1947). The work described here extends the scope of this type of investigation by determining the behaviour of filaments under conditions of steadily increasing strain. It was hoped in this way to separate the "elastic" and "plastic" components of strain, and so help to correlate the mechanical properties with the molecular re-arrangements which are known to take place when these substances are deformed.